THE PROPAGATION OF SONIC BANGS IN A NONHOMOGENEOUS STILL ATMOSPHERE

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ABSTRACT

The paper studies the propagation of sonic bangs in an atmosphere where the variation of the speed of sound with altitude causes the bang rays to be curved by refraction. Winds, which also lead to curvature of the rays, are not considered.

Phenomena of interest are "grazing," which occurs when a ray is tangential to the ground, and "focusing," which occurs when the rays have an envelope. Possible sonic bang patterns on the ground are studied, particularly those associated with acceleration to a supersonic cruising speed.

Finally some methods are discussed for adapting the well-established theory for the intensity of sonic bangs in a homogeneous atmosphere to an atmosphere whose properties vary with altitude.

INTRODUCTION

Sonic bangs are the subjective manifestations of the shock waves generated by an aircraft when it travels at a speed greater than that of sound. The disturbances imparted to the air by the motion of the aircraft form a series of "wavepackets," each of which is propagated along a "bang ray." Whitham [1] has shown that well away from the aircraft the disturbance pattern is relatively simple, and the wavepacket takes the form of an "N-wave." This is illustrated in Fig. 1. There is a sudden jump in pressure at the bow shock to a value greater than ambient, then a linear fall in pressure to a value less than ambient by an equal amount, then a sudden jump again at the stern shock, and finally a decay back to ambient conditions. Clearly this ideal N-wave is almost completely described by the "pressure jump" at the bow shock, and by the "duration of the positive phase."

A knowledge of the sonic bangs that will occur under given circumstances demands two things. First the bang rays along which the wavepackets are propagated must be constructed to determine where bangs will occur. Secondly, a study must be made of the variation of the pressure jump and of the duration of the positive phase as the wavepacket travels along a bang ray.

In this paper we study this problem for flight in an atmosphere whose properties vary with altitude. The variation of the speed of sound causes the bang rays to be curved by refraction. Study of this curvature leads to the concepts of "grazing" and of "focusing." In particular, a study is made of the ground pattern of events that occur when an aircraft accelerates to supersonic speed.

The atmosphere is assumed to be still and without dissipative properties. Accordingly winds, which can have an even more marked effect on the curvature of the bang rays than the variation in atmospheric properties, are not investigated. The main effect of the dissipative properties is to "soften" the sudden jumps in pressure, as illustrated in Fig. 2. Although the "rise time" thereby introduced is an important additional parameter, the dissipative properties do not greatly affect the pressure jump and the duration of the positive phase, which are the subject of study here.

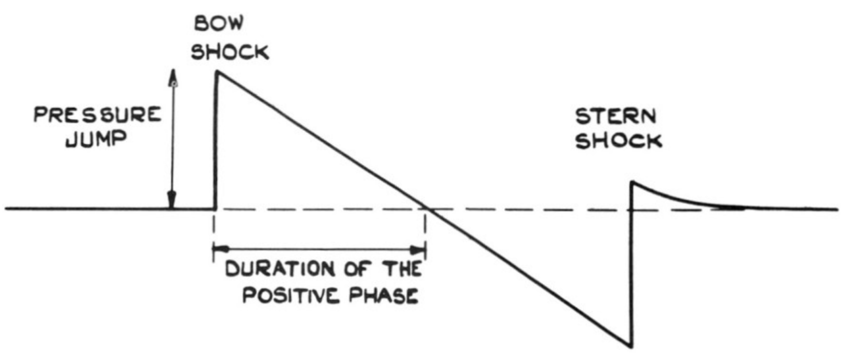

Figure 1. An indeal n-wave.

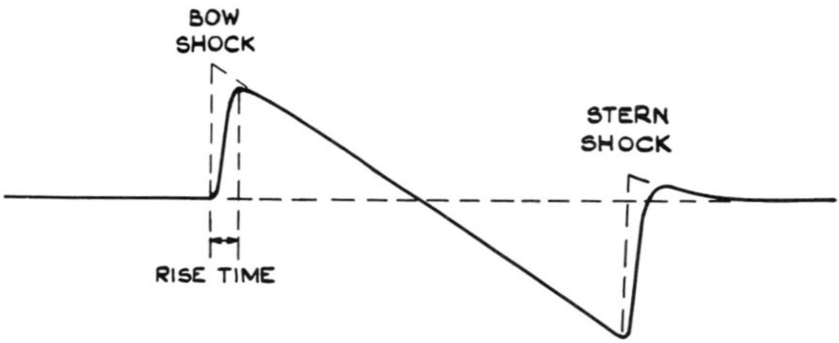

Figure 2. A "softened" n-wave.

THE OCCURRENCES OF BANGS

GENERAL CONSIDERATIONS

For the purpose of determining the occurrences of bangs an aircraft is treated as a point that moves along the flight path. The bang rays that emanate from a point of the flight path form part of a cone locally, but owing to variations in the ambient speed of sound in the atmosphere the bang rays are curved and in the large may be considered to form a "bang conoid," as illustrated in Fig. 3. Each point of the flight path is the apex of such a bang conoid.

In the present context the wavepackets are treated as points. The wavepackets generated at a given point of the flight path, for example P in Fig. 3, are propagated along the rays of the bang conoid through that point. The positions of all the wavepackets that have been generated up to a given time form a "wavefront." This is a conoid through the point, for example Q, representing the aircraft. For an atmosphere which in its undisturbed state is at rest the bang conoids and the wavefronts form two families of orthogonal surfaces.

The bang conoids associated with each point of the flight path usually have an envelope, which is the surface where the phenomenon of "focusing" occurs. There is a curve on each bang conoid which is its line of contact with the envelope. Each ray of a bang conoid meets this curve at the "point of focus" for that ray. Each wavefront also intersects the envelope in a curve. Along this curve the wavefront has a cusped edge, the wavefront being as illustrated by the two-dimensional example shown in Fig. 4.

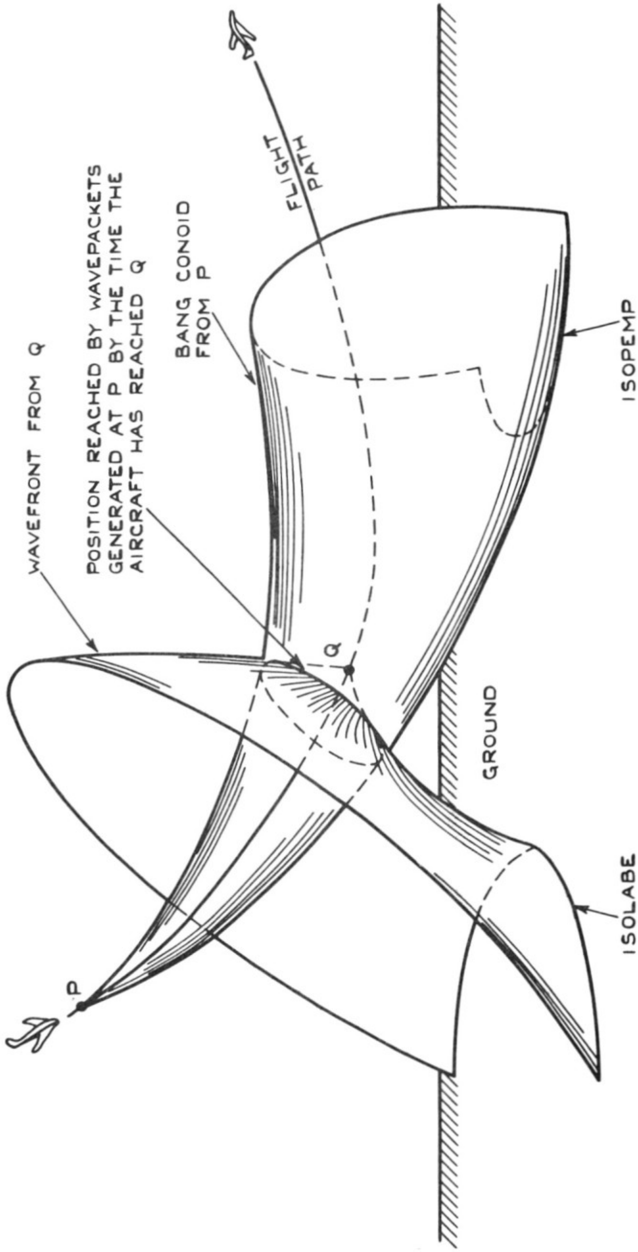

Figure 3. Bang conoids and wavefronts.

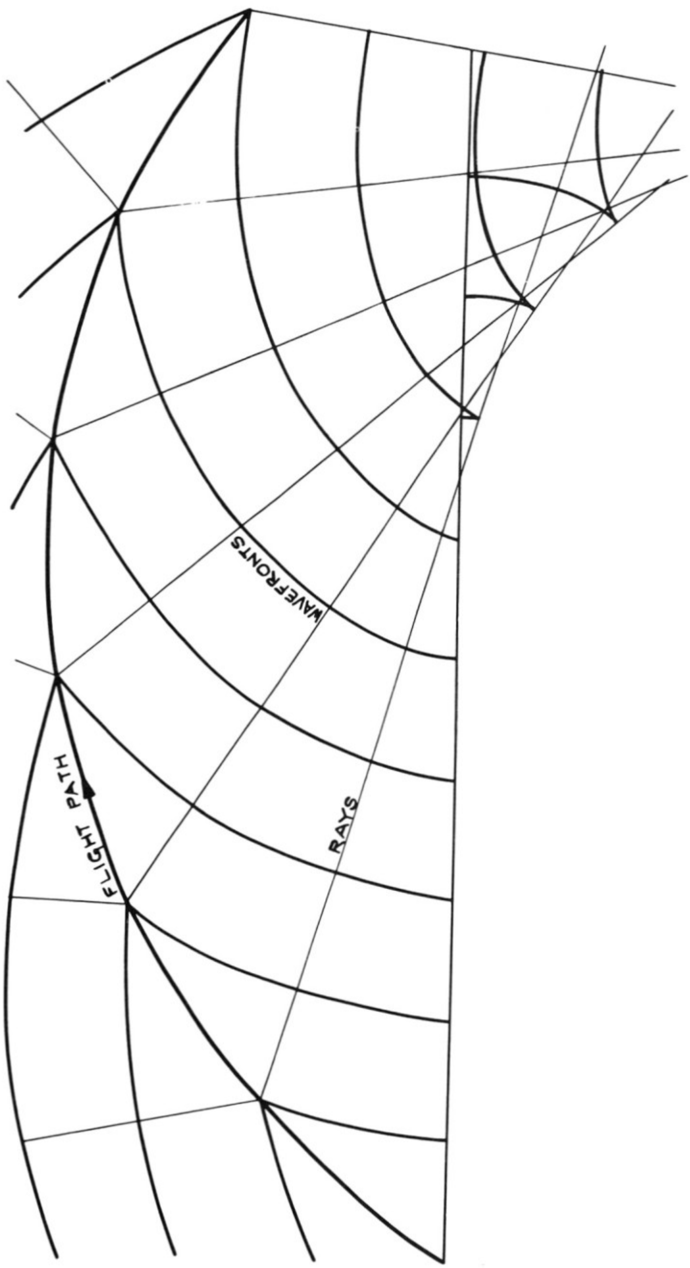

Figure 4. Focusing of a wavefront.

A bang conoid intersects the ground in a curve called an "isopemp," as shown in Fig. 3. Clearly an isopemp connects the points on the ground that receive bangs generated at the same instant of time. A wavefront intersects the ground in a curve called an "isolabe," as shown in Fig. 3. Clearly an isolabe connects the points on the ground where bangs are heard at the same instant of time. The envelope, or "focus surface," intersects the ground in a curve called a "ground focus line." The ground focus line is an envelope of the isopemps and a cusp-locus of the isolabes.

THE BANG RAYS AND BANG CONOID

Consider an atmosphere which in its undisturbed state is at rest and in which the speed of sound a is solely a function of altitude z. A bang ray will lie in a vertical plane, and the law of refraction of the bang ray is

$$D_{\varpi} r = \frac{a_a \sec \varphi}{a} \tag{1}$$

where r is the distance of a wavepacket along the bang ray from the apex of the bang conoid, ϖ is the associated horizontal distance, φ is the angle that the bang ray makes with the horizontal at the apex, a_a is the speed of sound at the apex, and D_{ϖ} is the differential operator with respect to ϖ.

Equation (1) may also be written in the forms

$$D_z r = \frac{a_a \sec \varphi}{\sqrt{a_a^2 \sec^2\varphi - a^2}} \tag{2}$$

$$D_z \varpi = \frac{a}{\sqrt{a_a^2 \sec^2\varphi - a^2}} \tag{3}$$

Moreover since a wavepacket is propagated at approximately the speed of sound, we have

$$D_t r = a$$

where t is time, which may be combined with Eq. (2) to yield

$$D_z t = \frac{a_a \sec \varphi}{a \sqrt{a_a^2 \sec^2\varphi - a^2}} \tag{4}$$

For the I.C.A.O. standard atmosphere the relation between speed of sound a and altitude z above sea level is given by

$$\left.\begin{aligned} a^2 &= a_g^2(1 - z/z_0) && \text{for } z \leqslant z_t \\ a &= a_t && \text{for } z \geqslant z_t \end{aligned}\right\} \tag{5}$$

where a_g = speed of sound at sea level (340.29 m/s)
a_t = speed of sound at the tropopause (295.07 m/s)
z_t = altitude of the tropopause (11,000 m)
z_t = altitude of the tropopause (11,000 m) and z_0 has the value 44,334 m

For this atmosphere Eqs. (2), (3), (4) can be integrated to yield the equations of a bang ray and the time $\Delta\tau$ for a wavepacket to travel along it. We obtain

$$\left.\begin{aligned} r &= -Z \operatorname{cosec}\varphi + z \operatorname{cosec}\varphi && \text{for } z_t \leqslant z,\ Z \\ r &= -Z \operatorname{cosec}\varphi + ZE_1\left(\frac{Z}{z_0}, \frac{z}{z_0}, \varphi\right) && \text{for } z,\ Z \leqslant z_t \\ r &= -Z \operatorname{cosec}\varphi + z_tE_1\left(\frac{z_t}{z_0}, \frac{z}{z_0}, \varphi\right) && \text{for } z \leqslant z_t \leqslant Z \end{aligned}\right\} \tag{6}$$

where

$$E_1(\eta,\zeta,\varphi) = \operatorname{cosec}\varphi - 2\,\frac{1-\eta}{\eta}\sec\varphi\left\{\tan\varphi \pm \sqrt{\sec^2\varphi - \frac{1-\zeta}{1-\eta}}\right\} \tag{7}$$

$$\left.\begin{aligned} \varpi &= -Z\cot\varphi + z\cot\varphi && \text{for } z_t \leqslant z,\ Z \\ \varpi &= -Z\cot\varphi + ZE_2\left(\frac{Z}{z_0}, \frac{z}{z_0}, \varphi\right) && \text{for } z,\ Z \leqslant z_t \\ \varpi &= -Z\cot\varphi + z_tE_2\left(\frac{z_t}{z_0}, \frac{z}{z_0}, \varphi\right) && \text{for } z \leqslant z_t \leqslant Z \end{aligned}\right\} \tag{8}$$

where

$$E_2(\eta,\zeta,\varphi) = \cot\varphi - \frac{1-\eta}{\eta}\left\{\tan\varphi + \varphi\sec^2\varphi \pm \left[\sqrt{\frac{1-\zeta}{1-\eta}\left(\sec^2\varphi - \frac{1-\zeta}{1-\eta}\right)} + \sec^2\varphi\cos^{-1}\left(\sqrt{\frac{1-\zeta}{1-\eta}}\cos\varphi\right)\right]\right\} \tag{9}$$

$$\left.\begin{aligned} a_a\,\Delta\tau &= -Z\operatorname{cosec}\varphi + z\operatorname{cosec}\varphi && \text{for } z_t \leqslant z, Z \\ a_a\,\Delta\tau &= -Z\operatorname{cosec}\varphi + ZE_3\left(\frac{Z}{z_0}, \frac{z}{z_0}, \varphi\right) && \text{for } z, Z \leqslant z_t \\ a_a\,\Delta\tau &= -Z\operatorname{cosec}\varphi + z_tE_3\left(\frac{z_t}{z_0}, \frac{z}{z_0}, \varphi\right) && \text{for } z \leqslant z_t \leqslant Z \end{aligned}\right\} \tag{10}$$

where

$$E_3(\eta,\zeta,\varphi) = \operatorname{cosec}\varphi - 2\,\frac{1-\eta}{\eta}\sec\varphi\left\{\varphi \pm \cos^{-1}\left(\sqrt{\frac{1-\zeta}{1-\eta}}\cos\varphi\right)\right\} \tag{11}$$

In these equations Z is the altitude of the apex of the bang conoid, and the constants of integration have been chosen so that $r = \varpi = \Delta\tau = 0$ at the apex. The positive signs apply to the point where the bang ray is downgoing at the altitude z, and the negative signs to the point where it is upgoing.

Now a bang ray makes an angle $\frac{1}{2}\pi - \mu$ with the direction of the flight path at its apex [2], where

$$\operatorname{cosec}\mu = M = V/a_a$$

and M is the Mach number of the aircraft at the apex and V is its speed. If the aircraft is climbing so that the flight path makes an angle Γ with the horizontal, then the angle φ is related to the angle Ω between the vertical plane containing the bang ray and the vertical plane containing the tangent to the flight path by the relation

$$\sin\mu = \cos\Gamma\cos\Omega\cos\varphi + \sin\Gamma\sin\varphi \tag{12}$$

Therefore the equation of the bang conoid in cylindrical polar coordinates ω, Ω, z with origin at the apex is obtained by eliminating φ between Eqs. (8) and (12). We shall need to express this equation in cartesian coordinates with origin at some arbitrary point on the track (the projection of the flight path on the ground). We shall derive this for the special case of flight in a straight line. We shall take the zx-plane to be the vertical plane through the flight path. Accordingly if x, y, z, are the cartesian coordinates of a point on a bang conoid, they are related to the cylindrical polar coordinates ϖ, Ω, z by the relations

$$x = \int \frac{Ma_a^2 \cos \Gamma}{\dot{V}} dM + \varpi \cos \Omega \tag{13}$$

$$y = \varpi \sin \Omega \tag{14}$$

where $\dot{V}$ is the acceleration of the aircraft. In terms of the same variables the altitude of the aircraft is given by

$$Z = \int \frac{Ma_a^2 \sin \Gamma}{\dot{V}} dM \tag{15}$$

GRAZING

Some typical isopemps are shown in Fig. 5. When the bang conoid just touches the ground the isopemp is merely the point of contact, G (Fig. 5(a)). Figure 5(b) shows the usual case when the bang conoid intersects the ground. The solid portion of the isopemp corresponds to downgoing rays. The broken portion corresponds to upgoing rays, which are not experienced in practice. The two portions join at the "points of graze," G, which are the points of contact of the tangents from the projection of the apex of the bang conoid, A. Figures 5(c) and 5(d) are examples of what can occur in diving flight.

The points of graze are of importance. They are given by the value of φ that makes the two values of $E_2(\eta,0,\varphi)$ in Eq. (9) equal, that is, by

$$\left.\begin{aligned} \sin \varphi &= \sqrt{\frac{Z}{z_0}} \qquad \text{for } Z \leqslant z_t \\ \sin \varphi &= \sqrt{\frac{z_t}{z_0}} \qquad \text{for } Z \geqslant z_t \end{aligned}\right\} \tag{16}$$

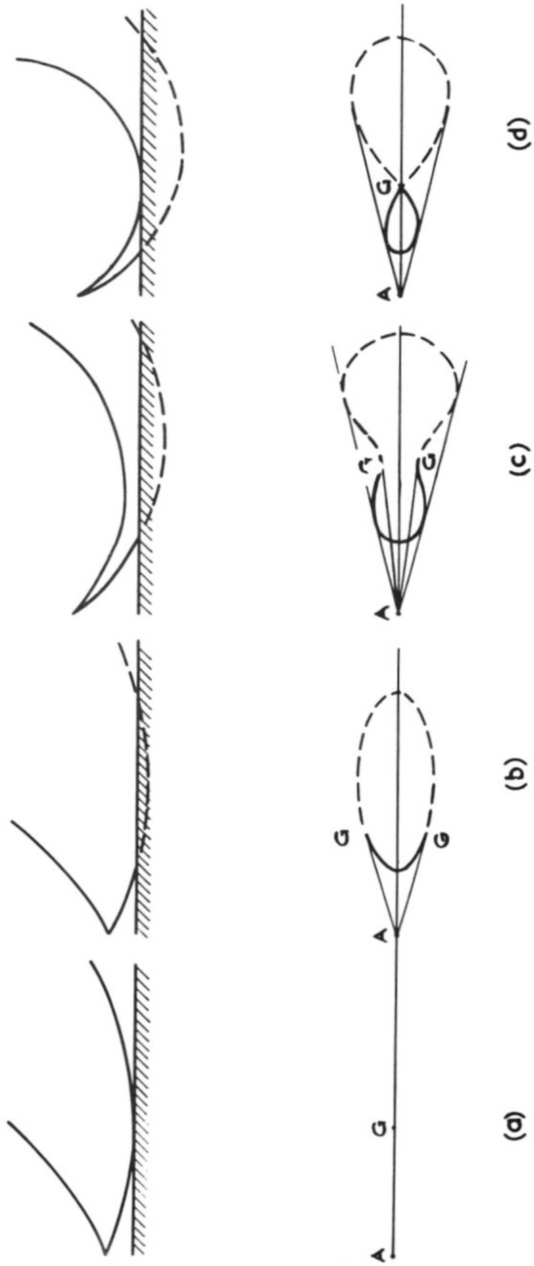

Figure 5. Some typical isotemps.

The relations (5) show that these two equations (16) may both be represented by the single relation

$$\cos\varphi = \frac{a_a}{a_g} \tag{17}$$

The locus of the points of graze associated with each point of the flight path is called the "graze line." For a given flight plan it is given in parametric form by Eqs. (13), (14), where ϖ is given by Eq. (8) with $z = 0$, Ω is given by Eq. (12), and φ is given by Eq. (17).

FOCUSING

The focus surface is the envelope of the bang conoids. It is found by expressing the condition that the bang conoids of adjacent points on the flight path intersect. The equation of the focus surface for the special case of flight in a straight line in the stratosphere is, therefore, the eliminant of dM, dZ, $d\varphi$, $d\varpi$, $d\Omega$ from the following five equations, which are derived from Eqs. (8), (12), (13), (14), (15).

$$d\varpi = -\cot\varphi\, dZ + \left[Z \operatorname{cosec}^2\varphi + z_t E_2'\left(\frac{z_t}{z_0}, \frac{z}{z_0}, \varphi\right)\right] d\varphi$$

$$-\frac{1}{M^2}\,dM = -\cos\Gamma\sin\Omega\cos\varphi\, d\Omega - (\cos\Gamma\cos\Omega\sin\varphi - \sin\Gamma\cos\varphi)\, d\varphi$$

$$0 = \frac{M a_a^2 \cos\Gamma}{\dot V}\,dM + \cos\Omega\, d\varpi - \varpi\sin\Omega\, d\Omega$$

$$0 = \sin\Omega\, d\omega + \varpi\cos\Omega\, d\Omega$$

$$dZ = \frac{M a_a^2\sin\Gamma}{\dot V}\,dM$$

where $E_2'(\eta,\zeta,\varphi)$ is the derivative of $E_2(\eta,\zeta,\varphi)$ with respect to φ, and is given by

$$E_2'(\eta,\zeta,\varphi) = -\operatorname{cosec}^2\varphi - 2\,\frac{1-\eta}{\eta}\sec^2\varphi\tan\varphi\left\{\cot\varphi + \varphi \pm\left[\sqrt{\frac{1-\zeta}{1-\eta}\Big/\left(\sec^2\varphi - \frac{1-\zeta}{1-\eta}\right)} + \cos^{-1}\left(\sqrt{\frac{1-\zeta}{1-\eta}}\cos\varphi\right)\right]\right\}$$

The eliminant is

$$\frac{\dot{V}Z}{a_a^2}\frac{\sec^2 \Gamma \sec^2 \Omega \operatorname{cosec} \varphi}{M^3} = \frac{\tan^2 \Omega \cot \varphi}{\varpi/Z} - \frac{(1 - \tan \Gamma \sec \Omega \cot \varphi)^2}{\operatorname{cosec}^2 \varphi + \left(\frac{z_t}{Z}\right) E_2'\left(\frac{z_t}{z_0}, \frac{z}{z_0}, \varphi\right)} \quad (18)$$

where ϖ, Ω, φ are given in terms of x, y and the flight variables by Eqs. (12), (13), (14).

The ground focus line is given by putting $z = 0$ in Eq. (18). A typical ground focus line, together with the associated isopemps, for an aircraft doing a straight and level acceleration to supersonic speed is shown in Fig. 6. For clarity the associated pattern of isolabes is shown separately in Fig. 7. Now in GRAZING it was shown that only part of each isopemp corresponds to downgoing rays: this part is represented by the full portion of the curve lying to the left of the point of contact with the graze line $GCBCG$ in Fig. 6. Correspondingly only part of the ground focus line $FCACF$ corresponds to downgoing rays: this part is the full portion of the curve CAC. The graze line, the focus line and the local isopemp (in this case that for $M = 1.22$) all touch at the points C. Further the boundary $GCACG$ of the area subjected to bangs (sometimes called the "cutoff line") is given by the focus line from A up to the points C, and by the graze line beyond the points C.

Focusing occurs, therefore, to a limited extent on the ground, represented by the portion of curve CAC in Fig. 6. The extent of this focusing is determined mainly by the location of the point on the track A and the points of graze C. The focused bangs received at these points are generated at Mach numbers which depend upon the instantaneous values of the aircraft's altitude, angle of climb and acceleration, and which are given by Eqs. (12), (18). The fore-and-aft distance between the point on the track A and the points of graze C can be derived from a knowledge of the "forward throw" of a bang, and the lateral distance between the points of graze C from a knowledge of the "lateral throw," or lateral distance from the track of a bang. These quantities too depend upon the instantaneous values of the aircraft's altitude, angle of climb and acceleration. The lateral extent d_l of the ground focus line is simply twice the lateral throw of the point of graze C. The fore-and-aft extent d_f depends upon the manner in which the aircraft accelerates over the distance Δx from the apex of the bang ray through A to the apex of the bang ray through C, and is given by

$$d_f = (\text{forward throw of } C) - (\text{forward throw of } A) + \Delta x$$

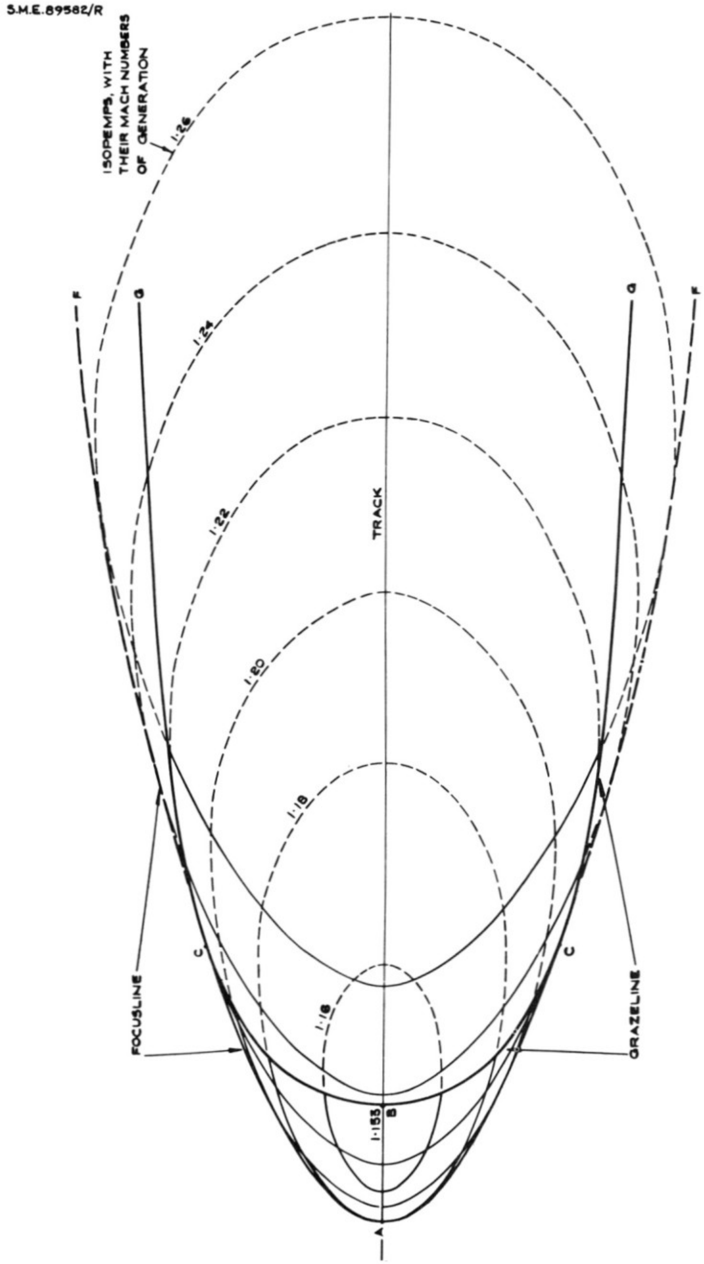

Figure 6. The pattern of isopemps, graze line and focus line for an aircraft during a straight and level acceleration to supersonic speed.

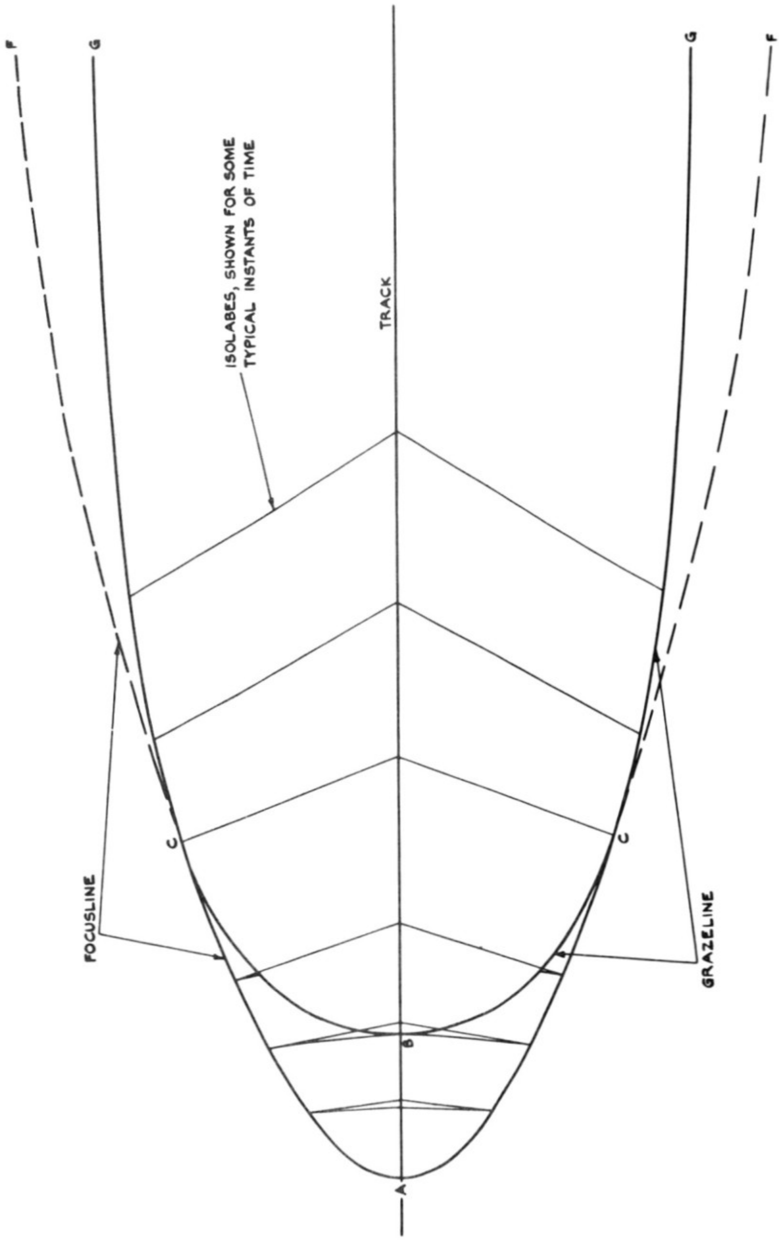

Figure 7. The pattern of isolabes, graze line and focus line for an aircraft during a straight and level acceleration to supersonic speed.

The manner in which the aircraft accelerates can be exhibited as a curve in what is effectively the velocity-acceleration plane, of which Fig. 8 is a diagrammatic example. A focused bang is first received when an aircraft crosses the curve for which a focused bang occurs at a point on the track. Throughout the shaded region focused bangs continue to be generated, until the aircraft crosses the curve for which focused bangs occur at the points of graze.

Two examples of acceleration path are shown in Fig. 8. *ABCD* represents a steady acceleration, and *A'B'CD* represents a "cutback" technique, designed to reduce the extent of focusing to a minimum. The fore-and-aft extent of the ground focus line d_f for these two examples is shown in Fig. 9, for zero angle of climb at an altitude of 13,200 *m* (43,300 ft). Although for steady acceleration the fore-and-aft extent of the ground focus line does not depend very much on the acceleration, Fig. 10 shows that the lateral extent d_l can be reduced appreciably by reducing the acceleration. For the cutback technique the fore-and-aft extent has been plotted in Fig. 9 against the acceleration after cutback, and the Mach number at which cutback is made is shown. The cutback technique appreciably reduces the fore-and-aft extent, and, moreover, the lower the Mach number at cutback the greater the reduction.

Unfortunately cutback is inconsistent with the need to get to cruising speed and altitude as quickly as possible. However, all that is really necessary is that the fore-and-aft acceleration be suddenly reduced: this can be achieved by suddenly increasing the angle of climb. For we have, approximately, that

$$\frac{T - D}{W} \simeq \frac{\dot{V}}{g} + \Gamma$$

$$= \frac{\dot{V}Z/a_a^2}{gZ/a_a^2} + \Gamma$$

where T = thrust of the engines
D = drag of the aircraft
W = weight of the aircraft
g = gravitational acceleration (9.81 m/s^2)

Clearly, therefore, a reduction in $\dot{V}Z/a_a^2$ can be achieved not only by cutting back the engines—that is by reducing $(T - D)/W$—but also by increasing the angle of climb Γ. Figure 11 shows the variation of the fore-and-aft extent of the ground focus line d_f, and also the amount of engine cutback $\Delta[(T - D)/W]$, as a function of the Mach number at cutback for

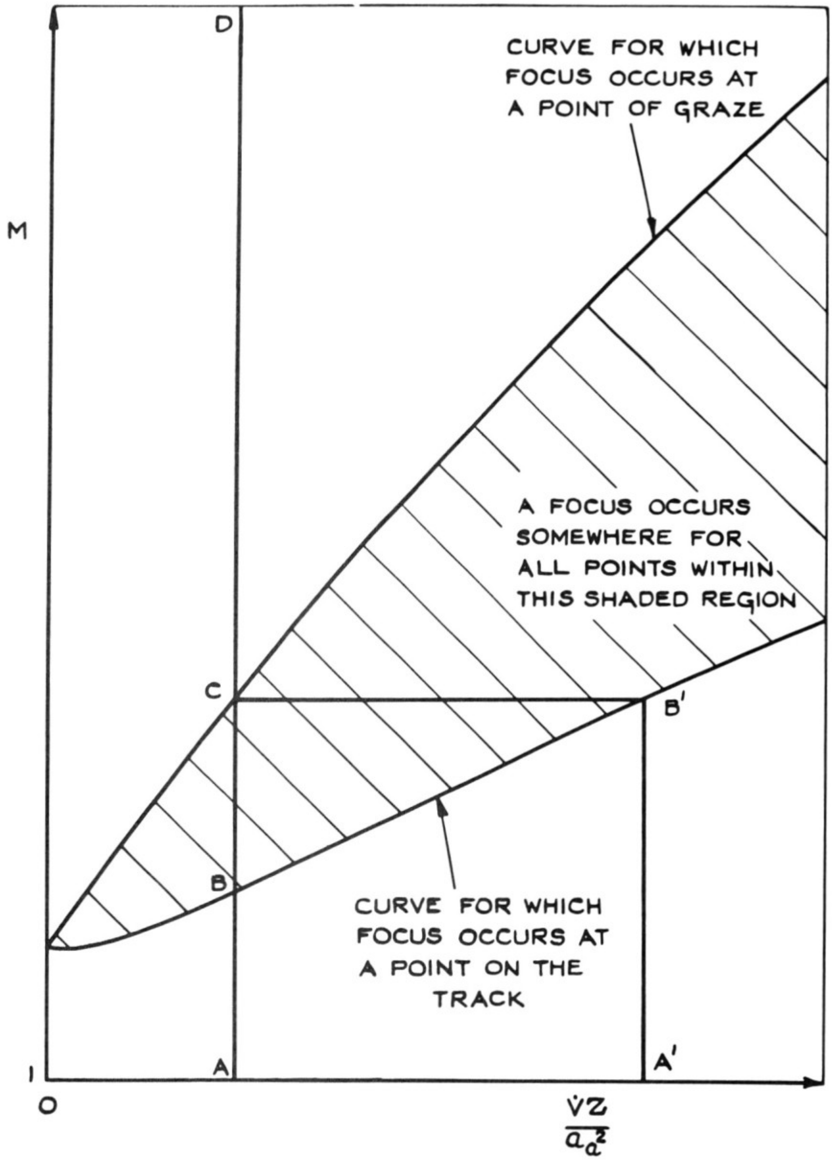

Figure 8. Diagram of different acceleration paths in the M, $\frac{\dot{V}Z}{a_a^2}$ plane.

the two cases of simple cutback and cutback involving an increase in the angle of climb. The difference in the fore-and-aft extent is small, but the cutback and climb technique leads to an appreciable reduction in the amount of engine cutback.

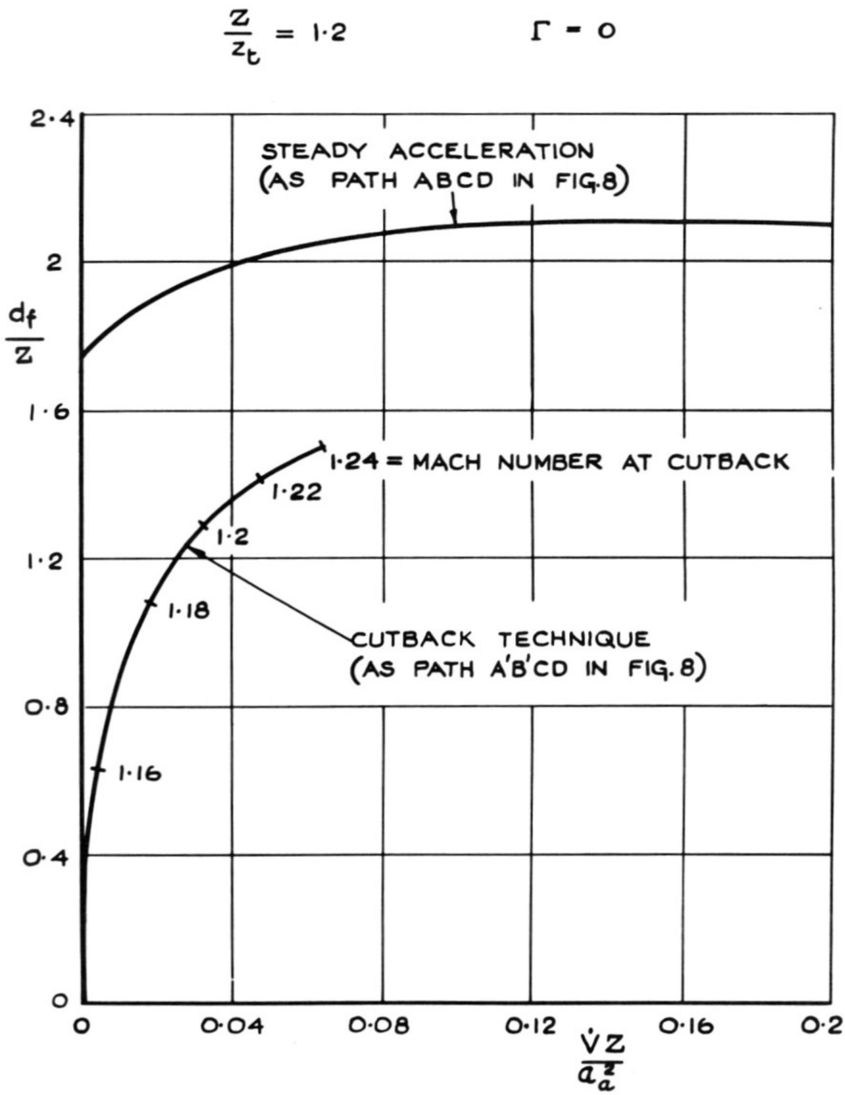

Figure 9. Fore-and-aft extent of the ground focus line.

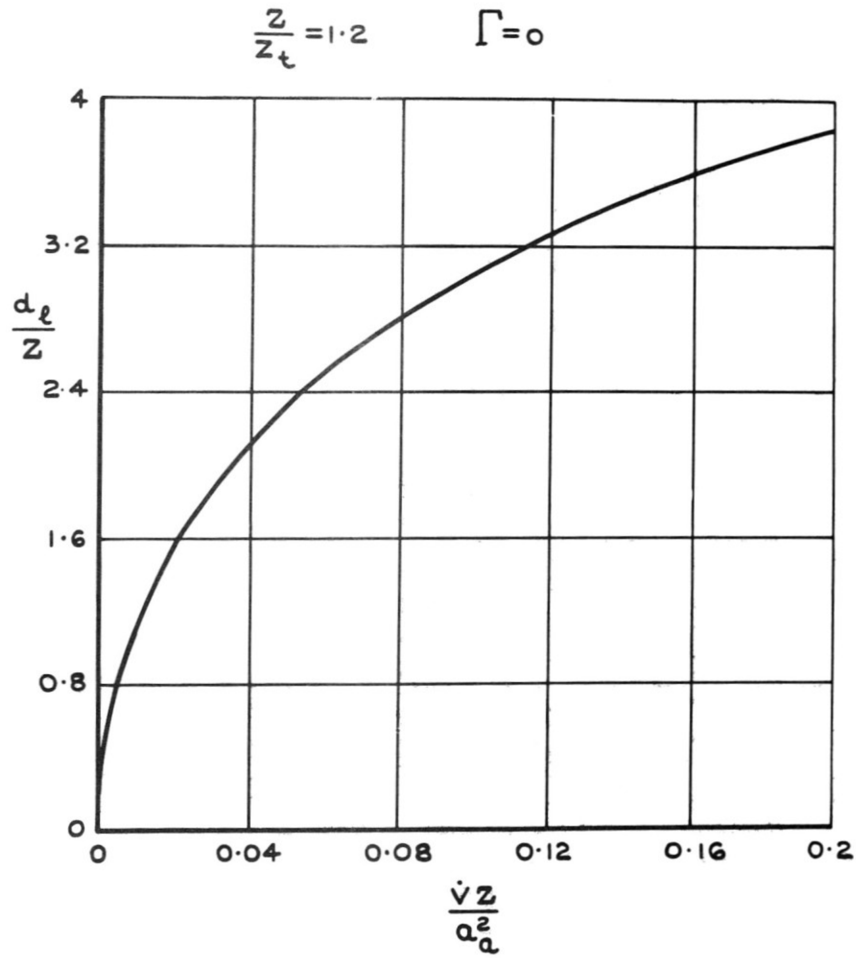

Figure 10. Lateral extent of the ground focus line.

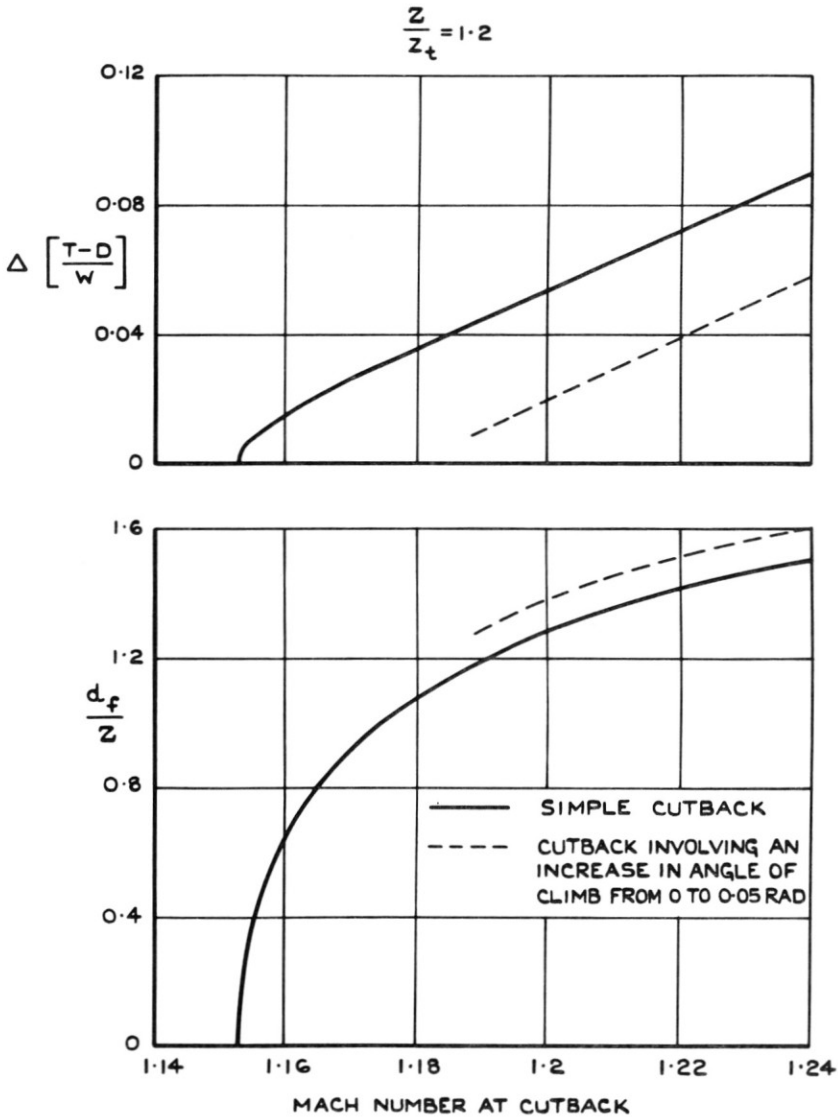

Figure 11. Effect of going into a climb at cutback.

THE INTENSITiES OF BANGS

FREE-AIR PREDICTIONS FOR A UNIFORM ATMOSPHERE

The free-air pressure jump and duration of the positive phase of a sonic bang have been derived rigorously only for the case of a uniform atmosphere. The classical formula [2], valid at large distances from the aircraft, are

$$\frac{\Delta P}{P_\infty} = \frac{2^{1/4}\gamma}{(\gamma + 1)^{1/2}} \frac{M^{3/4}}{(M^2 - 1)^{1/4}} \frac{\{\mathfrak{F}(\eta_0)\}^{1/2}}{r^{3/4}} K_f \tag{19}$$

$$a_\infty \, \Delta t = 2^{1/4} \, (\gamma + 1)^{1/2} \frac{M^{3/4}}{(M^2 - 1)^{1/4}} r^{1/4} \{\mathfrak{F}(\eta_0)\}^{1/2} K_f' \tag{20}$$

where ΔP = pressure jump
Δt = duration of the positive phase
P_∞ = ambient air pressure
a_∞ = ambient speed of sound
γ = air specific heats ratio (taken as 1.4)
M = aircraft Mach number
r = distance along a bang ray, and η_0 = value of η for which $\mathfrak{F}(\eta)$ is a maximum

$\mathfrak{F}(\eta)$ is given by

$$\mathfrak{F}(\eta) = \mathfrak{F}_v(\eta) - (M^2 - 1)^{1/2} \cos \vartheta \; \mathfrak{F}_l(\eta) \tag{21}$$

where

$$\mathfrak{F}_v(\eta) = \frac{1}{2\pi} \int_0^\eta S'(\xi) \frac{d\xi}{(\eta - \xi)^{1/2}} \tag{22}$$

$$\mathfrak{F}_l(\eta) = \frac{1}{2\pi} \int_0^\eta \frac{L'(\xi)}{\gamma P_\infty M^2} \frac{d\xi}{(\eta - \xi)^{1/2}} \tag{23}$$

$S(\xi)$ = cross-sectional area at station ξ from the nose
$L(\xi)$ = lift from the nose to station ξ
ϑ = angle that the plane containing the lift direction and the tangent to the flight path makes with the plane containing the bang ray and the tangent to the flight path: ϑ is acute when the lift has a positive resolute along the direction of the bang ray: in particular $\vartheta = \pi$ on the track for steady straight and level flight.

K_f and K_f' are so-called focus factors, and are given by

$$K_f = \left\{\frac{1 - r/r_f}{(r/r_f)^{1/2}} \int_0^{r/r_f} \frac{{}^1\!/_2\, d\lambda}{[\lambda(1 - \lambda)]^{1/2}}\right\}^{-1/2} \tag{24}$$

$$K_f' = \left\{\frac{1}{(r/r_f)^{1/2}} \int_0^{r/r_f} \frac{{}^1\!/_2\, d\lambda}{[\lambda(1 - \lambda)]^{1/2}}\right\}^{1/2} \tag{25}$$

where r_f is the distance along the bang ray to the point of focus on that ray. K_f and K_f' are shown as functions of the proportional distance along a ray towards the focus, r/r_f, in Fig. 12. According to Eq. (24), K_f tends to infinity as the focus is approached. However, the theory upon which this result is based then breaks down. Wavepackets propagated along neighbouring bang rays interfere when the rays converge. This modifies the wavepackets from the usual N-wave form shown in Fig. 1. However, some far from rigorous theoretical studies and some experimental results suggest that in the neighbourhood of the focus the maximum overpressure attained is of the order of that predicted by Eq. (19) with K_f having a value around 2.

EXTENSION TO AN ATMOSPHERE WHOSE PROPERTIES VARY WITH ALTITUDE

The properties of a real atmosphere, when still, vary with altitude only. There are no published theoretical methods for predicting the intensities of bangs for arbitrary flight in such an atmosphere. Empirical rules must therefore be employed. It is usually assumed [3] that the free-air pressure jump and duration of the positive phase for a real atmosphere are still given by Eqs. (19), (20), except that r is now the arc distance along a curved bang ray, and P_∞ is replaced by $\sqrt{P_g P_a}$ and a_∞ by a_g, where P_g and P_a are respectively the ambient air pressures at ground level and at the altitude of the aircraft when it generates the bang, and a_g is the ambient speed of sound at ground level. Specifically, it is assumed that the free-air pressure jump and duration of the positive phase are given by

$$\frac{\Delta P}{P_g} = \frac{2^{1/4}\gamma}{(\gamma + 1)^{1/2}} \frac{M^{3/4}}{(M^2 - 1)^{1/4}} \frac{\{\mathfrak{F}(\eta_0)\}^{1/2}}{r^{3/4}} \left(\frac{P_a}{P_g}\right)^{1/2} K_f \tag{26}$$

$$a_g\, \Delta t = 2^{1/4}(\gamma + 1)^{1/2} \frac{M^{3/4}}{(M^2 - 1)^{1/4}} r^{1/4} \{\mathfrak{F}(\eta_0)\}^{1/2} K_f' \tag{27}$$

but that in Eq. (23) for $\mathfrak{F}_l(\eta)$ P_∞ is replaced by P_a.

Guiraud's work [4] and many flight experiments suggest that any errors introduced by these empirical rules are not significant.

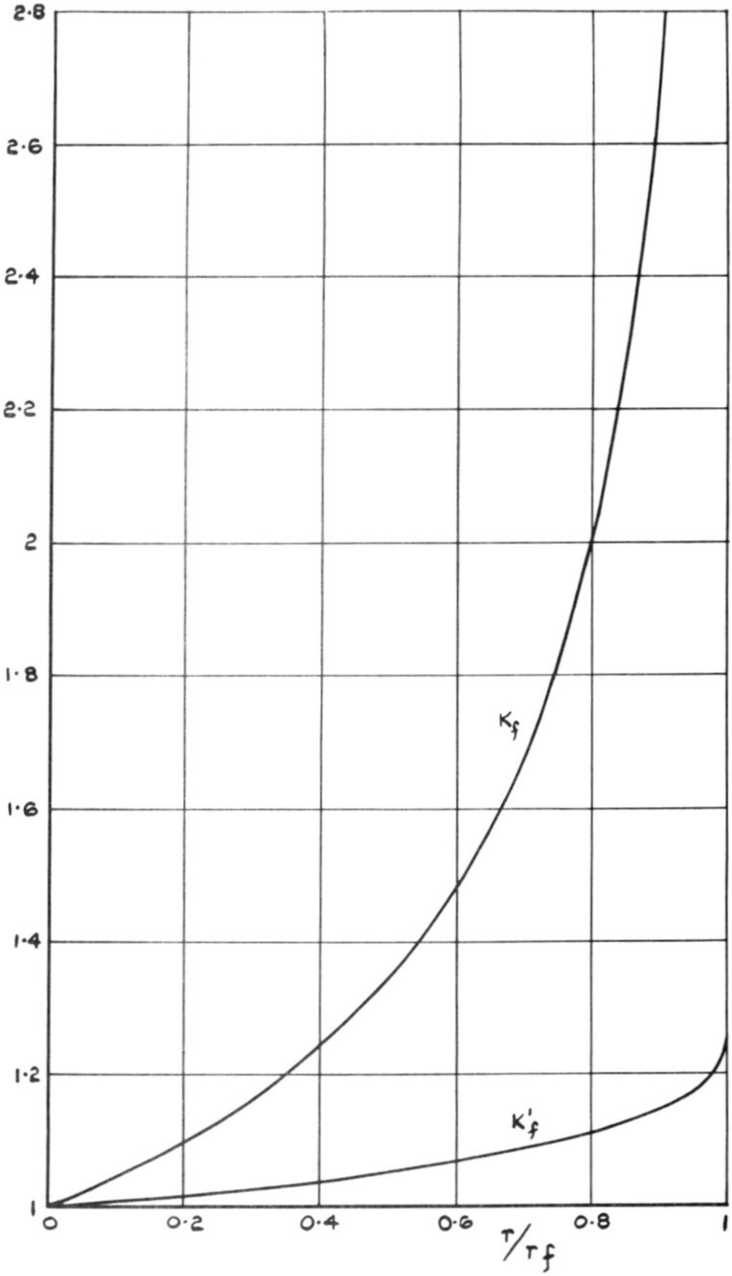

Figure 12. Focus factors.

EFFECTS NEAR THE GROUND

So far we have discussed the propagation of wavepackets under what are called free-air conditions, that is, away from ground and buildings. Near the ground and buildings the wave form is complicated by reflections and absorption.

Figure 13 shows the regular reflection that occurs when an N-wave is incident upon flat perfectly reflecting ground at an angle not near to grazing incidence. A recording instrument at D records two distinct N-waves, these being respectively the incident and reflected N-waves. This represents two double-bang events. At C the two N-waves abut, the bow shock of the reflected N-wave arriving simultaneously with the stern shock of the incident N-wave. This leads to a triple-bang event, the middle bang having twice the pressure jump of the other two. At B the incident and reflected N-waves overlap. This leads to a quadruple-bang event. At A, on the ground itself, reflection causes a single N-wave, owing to the superposition of the incident and reflected N-waves. The two bangs are of twice their free-air pressure jump.

When an N-wave is incident upon the angle formed by a wall and the ground the situation is still further complicated. From Fig. 14 it is easy to see that the waveform is made up of the overlapping of four N-waves, these being the N-waves that travel along the incident ray, and along the rays that undergo reflections from the ground alone, the wall alone, and from both the ground and the wall. This leads to a maximum of eight bangs in all. If an observer is so positioned that the four N-waves arrive simultaneously, then bangs of up to four times the free-air pressure jump would occur.

When an N-wave is incident upon flat perfectly reflecting ground at an angle near to grazing incidence Mach reflection may occur. The precise nature of the waveform pattern in this case is not yet fully understood, but it appears that the pressure jump experienced on the ground can be less than twice the incident free-air pressure jump associated with regular reflection. It may be as low as 1.5 times.

We have seen that the nature of the waveform, and hence the intensities of bangs, in the neighbourhood of ground and buildings depends upon the position of an observer relative to the ground and buildings, and upon the angle of incidence of the wavepacket. Nevertheless a rough attempt is usually made to assess the maximum overpressure likely to be attained by introducing a further factor K_r, called a reflection factor, to the right-hand side of Eq. (26). Indeed, the free-air intensity is somewhat artificial, for it does not apply in conditions that are commonly met. Accordingly it has become customary to quote intensities for a condition "near the ground in

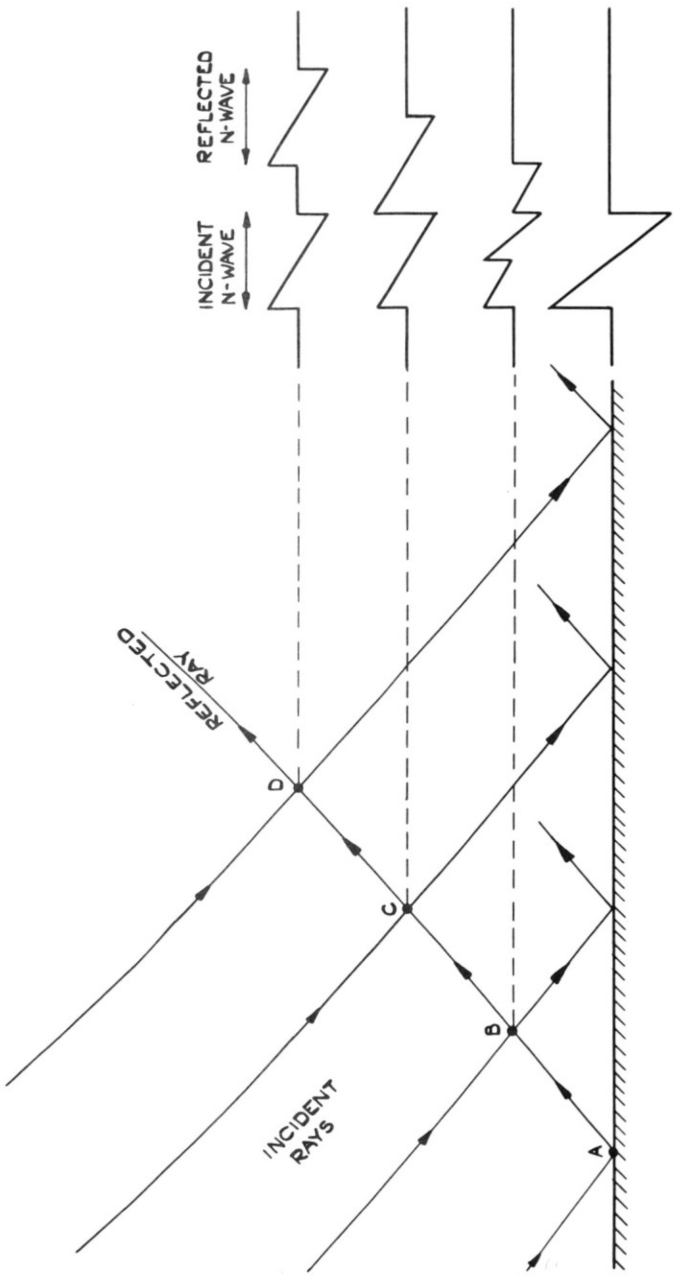

Figure 13. Impingement of an n-wave on flat ground.

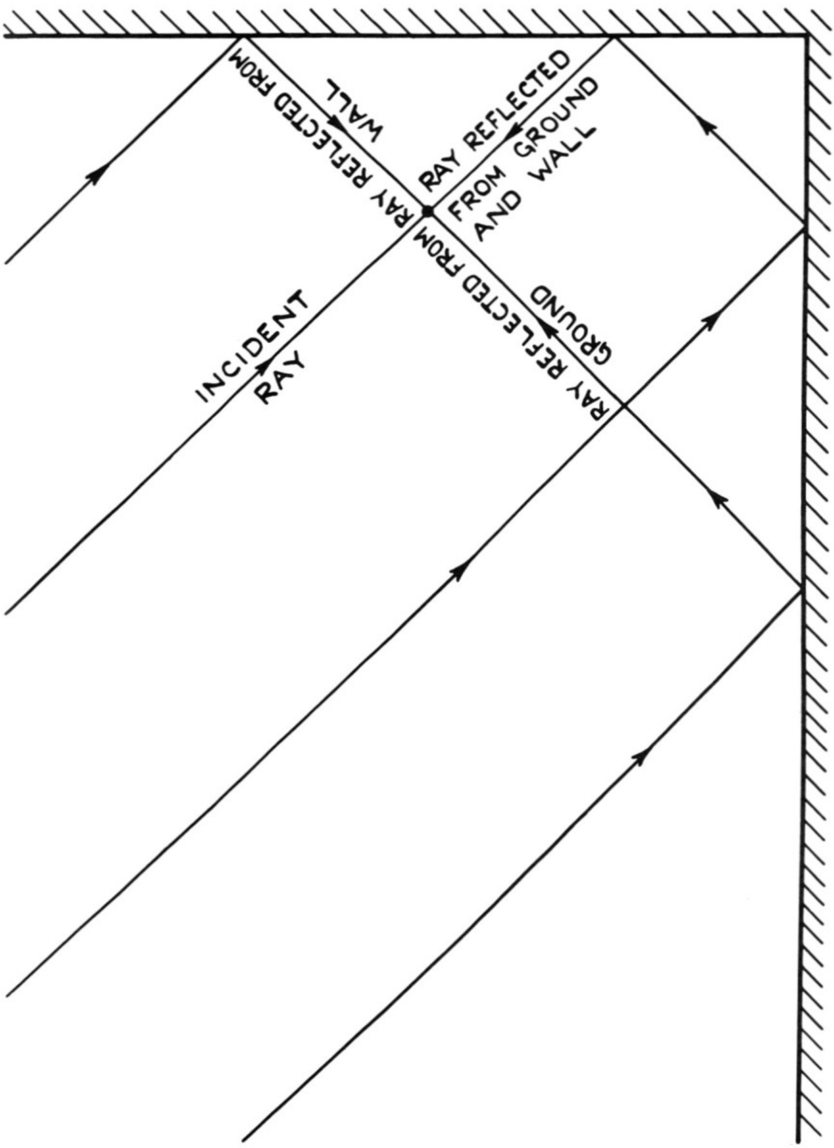

Figure 14. Impingement of an n-wave on the angle formed by a wall and the ground.

the open," for this is closely the condition experienced on the ears of an observer standing on flat open ground away from buildings, etc. It corresponds to point A in Fig. 13. Moreover, as a standard, in the absence of knowledge of the nature of the ground, or of the angle of incidence of the wavepacket, it is usual to assume that the ground is perfectly reflecting, and the incidence not near grazing. As we have seen this leads to a reflection factor K_r of 2 in determining the pressure jump near the ground in the open.

EFFECTS OF AIRCRAFT SHAPE AND LIFT

The formulae given by Eqs. (26), (27) depend upon the value of $\mathfrak{F}(\eta_0)$. If the aircraft has negligible lift $\mathfrak{F}(\eta_0)$ can be approximated by $\mathfrak{F}_v(\eta_0)$, which, by Eq. (22), depends on the volume distribution only. We can write

$$\mathfrak{F}_v(\eta_0) = K_v^2 \frac{S_{\max}^{3/2}}{v^{1/2}} \tag{28}$$

where $S_{\max}$ is the maximum cross-sectional area of the aircraft, v is its volume, and K_v is a "volume distribution factor." Its value depends upon how the volume of the aircraft is distributed, but is usually about 0.55.

When the aircraft has appreciable lift we must use $\mathfrak{F}(\eta_0)$ itself. Equations (21), (22), (23) show that this depends not only on the distribution of lift as well as of volume, but also upon the Mach number of the aircraft and the position of the observer relative to its track. We shall consider the case in which the aircraft has sufficient lift for $\mathfrak{F}(\eta_0)$ to be dominated by $\mathfrak{F}_l(\eta_0)$. If the lift is borne mainly by a wing of length c, Eq. (23) (with P_∞ replaced by P_a) shows that

$$\begin{aligned} \mathfrak{F}(\eta_0) &= -\ (M^2 - 1)^{1/2} \cos\vartheta \mathfrak{F}_l(\eta_0) \\ &= -\ K_l^2 \frac{(M^2 - 1)^{1/2}}{M^2} \frac{\cos\vartheta}{\gamma P_a} \frac{W}{c^{1/2}} \end{aligned} \tag{29}$$

where W is the weight of the aircraft, and K_l is a "lift distribution factor." Its value depends upon how the lift is distributed, but is usually about 0.6.

SYMBOLS

a	speed of sound
a_a	speed of sound at the altitude of the aircraft
a_g	speed of sound at sea level
a_t	speed of sound at the tropopause
a_∞	speed of sound in a uniform atmosphere
c	length of wing over which the lift is borne
D	drag of the aircraft
d_f	fore-and-aft extent of the ground focus line
d_l	lateral extent of the ground focus line
$E_1(\eta,\zeta,\varphi)$	function defined by Eq. (7)
$E_2(\eta,\zeta,\varphi)$	function defined by Eq. (9)
$E_2'(\eta,\zeta,\varphi)$	derivative of $E_2(\eta,\zeta,\varphi)$ with respect to φ
$E_3(\eta,\zeta,\varphi)$	function defined by Eq. (11)
$\mathfrak{F}(\eta)$	function defined by Eq. (21)
$\mathfrak{F}_l(\eta)$	function defined by Eq. (23)
$\mathfrak{F}_v(\eta)$	function defined by Eq. (22)
g	gravitational acceleration
K_f	focus factor for pressure jump, defined by Eq. (24)
K_f'	focus factor for positive duration, defined by Eq. (25)
K_l	lift distribution factor, defined by Eq. (29)
K_r	reflection factor
K_v	volume distribution factor, defined by Eq. (28)
$L(\xi)$	lift from the nose to station ξ
M	Mach number of the aircraft
P_a	ambient pressure at the altitude of the aircraft
P_g	ambient pressure at sea level
P_∞	ambient pressure in a uniform atmosphere
ΔP	pressure jump
r	distance along a ray from the aircraft
r_f	distance along a ray from the aircraft to the point of focus
$S(\xi)$	cross-sectional area at station ξ from the nose
$S_{\max}$	maximum cross-sectional area
T	thrust of the engines
t	time
Δt	duration of the positive phase
V	speed of the aircraft
$\dot{V}$	forward acceleration of the aircraft
v	volume of the aircraft
W	weight of the aircraft
x	horizontal distance along the track

Δx	length of focus-generating portion of the flight path
y	horizontal distance from, and at right angles to, the track
Z	altitude of the aircraft
z	altitude
z_t	altitude of the tropopause
z_0	altitude defined by Eq. (5)
Γ	angle of climb
γ	air specific heats ratio
η_0	value of η for which $\mathfrak{F}(\eta)$ is a maximum
ϑ	angle that the plane containing the lift direction and the tangent to the flight path makes with the plane containing the bang ray and the tangent to the flight path: ϑ is acute when the lift has a positive resolute along the direction of the bang ray: in particular $\vartheta = \pi$ on the track for steady straight and level flight
μ	Mach angle of the aircraft
ϖ	horizontal distance from the aircraft
$\Delta\tau$	time taken for a wavepacket to travel along a bang ray
φ	angle that a bang ray makes with the horizontal at its origin
Ω	angle between the vertical plane containing a bang ray and the vertical plane containing the tangent to the flight path

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COMMENTARY

J. P. GUIRAUD (*O.N.E.R.A., Chatillons Bagneux, France*): Le Docteur Warren a mentionné dans sa conférence le problème de l'intensité du bang au voisinage de la focalisation. C'est un très difficile problème mais je voudrais signaler qu'il est possible d'élaborer un commencement de théorie rationnellement fondée. Un peu de réflexion montre que le problème est linéaire parce qu'il est local et que l'on n'a pas à prendre en compte le phénomène d'accumulation qui contribue à construire l'onde en N hors focalisation. Ou peut alors avoir recours à une théorie acoustique. Je ne peux ici qu'indiquer les résultats de l'étude. Si ϵ désigne la surpression relative $\Delta p/p_0$ hors focalisation, la surpression relative au voisinage de la caustique cot $O(\epsilon^{5/6})$ ce qui indique que l'on est en présence d'un coefficient d'amplification $O(\epsilon^{1/6})$. On trouve également que ce coefficient d'amplification varie comme $R^{-1/3}$ si R est le rayon de courbure de la surface caustique, s'agissant en fait de la section par un plan normal parsant par le rayon sonore.

COMMENTARY

G. H. LEE (*Handley Page, Ltd., London*): The paper shows that the bang is reduced if acceleration is reduced and if transition to supersonic speed occurs in a climb. Has Mr. Warren studied the effect of increasing the angle of climb beyond the 3° quoted in the paper? Clearly increasing the height at which you go transsonic will reduce the bang and if steep climbs also help, it seems that the requirement for reduced bang may lead to rather large engines and perhaps big wing areas. This would suggest an aeroplane capable of flying at a fairly high Mach number. Perhaps, therefore, the American concept of the M-3.0 SST may be less offensive on the ground than the Anglo-French M-2.0 or 2¼ aeroplane. I would like to hear comments from designers on this. Has Mr. Warren made any assessment of a possible weight penalty due to extra engine size and weight?

COMMENTARY

Y. TAUB (*Technical University, Delft, Netherlands*): I should like to ask Mr. Warren whether he could tell us something on the influence of the "cutback" method on the fuel consumption. The fuel consumption may increase appreciably during the transition period.

COMMENTARY

H. A. GOLDSMITH (*British Aircraft Corporation (Operating) Ltd., Bristol, England*): I would like to reply to Mr. Lee. The essence of Mr. Warren's "cutback" technique is the use of a low value of the longitudinal acceleration in the critical speed range. The climb is only one possible way of achieving this and does not in itself contribute very significantly to the reduction in boom levels. Thus there is no design pressure to overengine the aircraft for this reason.

The low acceleration climbs are certainly not optimum from a performance point of view, but the actual fuel penalties involved are in fact quite small.

REPLY

In reply to Mr. Lee, the paper shows that the extent of the area on the ground subjected to focused bangs is reduced if acceleration is reduced, especially if it is reduced in the manner referred to as the cutback technique. And, of course, one way of reducing the acceleration without reducing power is to go into a climb. I have not made specific calculations other than those described in the paper. In regard to Mr. Lee's later points about the type of aircraft likely to be least offensive from the sonic bang point-of-view, I must point out that the area subjected to focused bangs, with which I have dealt primarily, is only part of the story. The intensities of the bangs is also important, and, as Mr. Goldsmith has indicated, these are not reduced significantly by the techniques proposed. Indeed, putting in large engines and putting on a bigger wing in order to be able to make the acceleration to cruising speed in a steeper climb and at a greater altitude would not necessarily lead to less intense bangs if they led to a larger aircraft.

In reply to Mr. Taub, there is of course some penalty in fuel consumption associated with the cutback technique, especially if it is done by reducing power, but the calculations that I have done support Mr. Goldsmith's contention that the penalties are in fact quite small.